1 Graphs

Definition. *A graph G is a finite nonempty set, V(G), of objects, called vertices, together with a set, E(G), of unordered pairs of distinct vertices. The elements of E(G) are called Edges.*

Terminologies 1

- $e = u, v \in E(G)$
- we say that u and v are **adjacent** vertices
- e is **incident** with vertices u and v.
- e **joins** u and v
- vertex adjacent to vertex u are called **neighbours** of u. The set of neighbours of u is denoted by $N(u)$

Definition. *Two graphs G_1 and G_2 are **isomorphic** if there exist a bijection $f : V(G_1) \to V(G_2)$ such that f(u) and f(v) are adjacent in G_2 if and only if u and v are adjacent in G_1 Remember to modify the information above.*

Definition. *The number of edges incident with a vertex v is called the **degree** of v*

Theorem. *For any graph G we have*

$$\sum_{v \in V(G)} deg(v) = 2|E(G)|$$

*also known as **Handshaking Lemma***

Corollary. *The number of vertices of odd degree in a graph is even.*

Corollary. *The average degree of a vertex in the graph H is*

$$\frac{2|E(G)|}{|v(G)|}$$

Terminologies 2

A graph in which every vertex has degree k, for some fixed k, is called a **k-regular** graph.

Definition. *A **Complete graph** is one in which all pairs of distinct vertices are adjacent. The complete graph with p vertices is denoted by $K_p, p \geq 1$*

2 Bipartite

Definition. *A graph in which the vertices can be partitioned into two sets A and B, so that all edges join a vertex in A to a vertex in B, is called a **bipartite graph**, with bipartition(A,B)*

Terminologies 3

The **complete bipartite graph** $K_{m,n}$ has all vertices in A adjacent to all vertices in B, with $|A| = m, |B| = n$

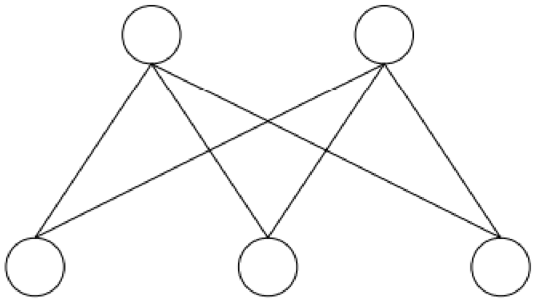

Figure 4.11: The complete bipartite graph $K_{2,3}$

Definition. *For $n \geq 0$, the n-cube is the graph whose vertices are the $\{0, 1\}$ strings of length n, and two strings are adjacent if and only if they differ in exactly one position.*

Terminologies 4

- |V| of n-cube is 2^n
- |E| of n-cube is $n \times 2^{n-1}$
- n-cube is bipartite
- n-cube is connected

3 Characterizing bipartite graph

Lemma. *An odd cycle is not bipartite.*

Theorem. *A graph is bipartite if and only if it has no odd cycles.*

4 Walks and Paths

Definition. *A **Subgraph** of a graph G is a graph whose vertex set is a subset U of $V(G)$ and whose edge set is a subset of those edges of G that have both vertices in U.*

spanning subgraph is graph with all vertices but not all edges.
proper subgraph is graph which is not equal to G.

Theorem. *if there is a walk from vertex x to vertex y in G, then there is a path from x to y*

Corollary. *let x, y, z be vertices of G. If there is a path from x to y in G and a path from y to z, then there is a path from x to z in G.*

5 Cycles

Definition. *A **cycle** is a connected graph that is regular of degree 2.*

Definition. *Alternate **Path**: The subgraph we get from a cycle by deleting one edge is called a path.*

Terminologies 5

- A cycle with n edges is called an n-cycle or a cycle of length n.
- Shortest possible cycle in a graph is a 3-cycle.
- A spanning cycle in a graph is known as a Hamilton Cycle.

Theorem. *If every vertex in G has degree at least 2, then G contains a cycle.*

Definition. *The **girth** of a graph G is the length of the shortest cycle in G.*

Connected

Definition. *A graph G is **connected** if, for each two vertices x and y, there is a path from x to y.*

Theorem. *Let G be a graph and let v be a vertex in G. If for each vertex w in G there is path from v to w in G, then G is connected.*

Definition. *A **Component** of G is a subgraph C of G such that*

a) C is connected

b) No subgraph of g that properly contains C is connected.

6 Cuts

Definition. *Given a graph G and $X \subseteq V(G)$, let $\delta_G(X)$ denote the **cut** of X in G, meaning the set of edges of G with one endpoint in X and one endpoint in $V(G) \setminus X$.*

Theorem. *A graph G is not connected if and only if there exist a proper subset of $V(G)$ such that the cut induced by X is empty.*

7 Eularian Circuit

Definition. *An **Eularian Circuit** of a graph G is a closed walk that contains every edge of G exactly once.*

Theorem. *Let G be a connected graph, then G has an Eularian Circuit if and only if every vertex has even degree.*

Definition. *An edge e of G is a **Bridge** if G-e (G\e) has more components than G*

Lemma. *if $e = \{x, y\}$ is a bridge of a connected graph, then $g - e(g\backslash e)$ has precisely two components; furthermore x and y are in two different components.*

Theorem. *An edge e is a bridge of a graph if and only if it is not contained in any cycle of G.*

Corollary. *If there are two distinct paths from vertex u to vertex v in G, then G contains a cycle.*

Terminologies 6
If graph G has no cycles, then each pair of vertices is joined by at most one path.

8 Tree

Definition. *A **tree** is a connected graph with no cycles.*

Definition. *A **forest** is a graph with no cycles.*

Terminologies 7
Every tree is a forest but every forest is not a tree.

Lemma. *if u and v are vertices in a tree T, then there is a unique u, v-path in T.*

Lemma. *Every edge of tree T is a bridge*

Theorem. *If T is a tree, then*

$$|E(T)| = |V(T)| - 1$$

Corollary. *If G is a forest with k components, then*

$$|E(T)| = |V(T)| - k$$

Definition. *A **leaf** in a tree is a vertex of degree 1.*

Theorem. *A tree with at least two vertices has atleast two leaves.*

Terminologies 8

- The number of leaves (vertex of degree 1) in a tree is given by

$$n_1 = 2 + \sum_{r \geq 3} (r-2) n_r$$

where r is the degree of vertex, and n_r is the number of vertex of degree r.

- A tree that contains a vertex of degree r has at least r vertices of degree one.

9 Spanning trees

Definition. *A spanning subgraph which is also a tree is called a **spanning tree**.*

Theorem. *A graph G is connected if and only if it has a spanning tree.*

Corollary. *I G is connected with p vertices and $q = p - 1$ edges, then G is tree.*

Theorem. *If T is a spanning tree of G and e is an edge not in T, then $T + e$ contains exactly one cycle C. Moreover if e' is any edge on C, then $T + e - e'$ is also a spanning tree of G.*

Theorem. *If T is a spanning tree of G and e is an edge in T, then $T - e$ has 2 components. If e' is in cut induced by one of the components, then T - e + e' is also a spanning tree of G.*

10 Planar Graphs

Definition. *A graph G is **planar** if it has a drawing in the plane so that its edges intersect only at their ends, and so that no two vertices coincide. The actual drawing is called **Planar Embedding** of G.*

Terminologies 9

- A graph is planar if and only if each of it's component is also planar
- A planar embedding partition the plane into connected regions called faces
- the outer face in the planar embedding is unbounded.
- the subgraph formed by the vertices and edges in a face is called **boundary** of the face.
- Two faces are **adjacent** if they are incident with a common edge
- Boundary Walk

- The number of edges in the boundary walk of face f is called the **degree** of the face f.
- A bridge of a planar embedding is always incident with just one face.
- A bridge of a planar embedding is always contained in the boundary walk twice, one for each side.
- bridge contributes 2 to the degree of the face with which it is incident.
- Every edge of a cycle in incident with exactly two faces, and is contained in the boundary walk of each face precisely once.
- every edge in a tree is a bridge. so the planar embedding of a tree T has a single face/

Theorem. *if we have a planar embedding of a connected graph G with faces* $f_1, \ldots, f_s$*, then*

$$\sum_{i=1}^{s} deg(f_i) = 2|E(G)|$$

Also known as ***Face Shaking Lemma***

Corollary. *If the connected graph G has a planar embedding with f faces, the average degree of a face in the embedding is* $\frac{2|E(G)|}{f}$

10.1 Euler's Formula

Theorem. *Let G be a connected graph with p vertices and q edges. If G has a planar embedding with f faces, then*

$$p - q + f = 2$$

Terminologies 10

For a given planar graph, the number of faces is always the same, regardless of how you draw it.

10.2 Nonplanar Graphs

Lemma. *If G contains a cycle, then in a planar embedding of G, the boundary of each face contains a cycle*

Lemma. *Let G be a planar embedding with p vertices and q edges. If each face of G has degree at least d*, then* $(d * -2)q \leq d * (p - 2)$

Theorem. *In a planar graph G with* $p \geq 3$ *vertices and q edges, we have*

$$q \leq 3p - 6$$

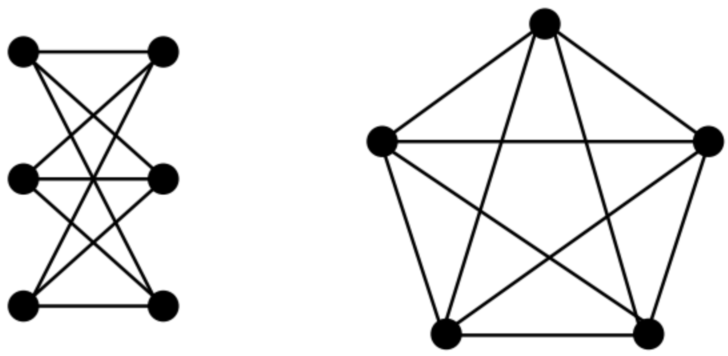

Figure 1: $K_{3,3}$ and K_5

Corollary. *K_5 is not planar*

Corollary. *A planar graph has a vertex of degree at most five.*
this only means atleast one vertex has degree less than equal to 5

Theorem. *In a bipartite planar graph G with $p \geq 3$ vertices and q edges, we have*

$$q \leq 2p - 4$$

Lemma. *$K_{3,3}$ is not planar.*

10.3 Kuratowski's Theoram

Definition. *An **Edge subdivision** of a graph G, is obtained by applying the following operation, independently, to each edge of G. replace the edge by a path of length 1 or more, if the path has length $m > 1$, then there are $m - 1$ new vertices, and $m - 1$ new edges created, if the path has lenght $m = 1$, then the edge is unchanged.*

Theorem. *A graph in not planar if and only if it has a subgraph that is an edge subdivision of K_5 or $K_{3,3}$*

11 Colouring and Planar Graph

Definition. *A k-colouring of a graph G, is a function from $V(G)$ to a set of size k (whose elements are called colours), so that adjacent vertices always have different colours. A graph with a k-colouring is called **k-colourable** graph*

Theorem. *A graph is 2-colourable if and only if it is bipartite.*

Theorem. *K_n is n-colourable, and not k-colourable for any $k < n$*

Theorem. *Every planar graph is all 4,5,6-colourable.* ***Read Proof***