Theorem Sheet Math 136

4.5

A matrix that can be obtained by performing a *single* ERO on the identity matrix is called an **elementary matrix**.

Proposition 4.5.3 Let $A \in M_{m\times n}(\mathbb{F})$ and suppose that a single ERO is performed on it to produce matrix B. Suppose, also, that we perform the same ERO on the matrix I_m to produce the elementary matrix E. Then

$$B = EA.$$

Corollary 4.5.4 Let $A \in M_{m\times n}(\mathbb{F})$ and suppose that a finite number of EROs, numbered 1 through k, are performed on A to produce a matrix B. Let E_i denote the elementary matrix corresponding to the ith ERO $(1 \le i \le k)$ applied to I_m. Then

$$B = E_k \dots E_2 E_1 A.$$

4.6

Definition 4.6.1
Invertible Matrix
We say that an $n \times n$ matrix A is **invertible** if there exist $n \times n$ matrices B and C such that $AB = CA = I_n$.

Proposition 4.6.2 **(Equality of Left and Right Inverses)**

Let $A \in M_{n\times n}(\mathbb{F})$. If there exist matrices B and C in $M_{n\times n}(\mathbb{F})$ such that $AB = CA = I_n$, then $B = C$.

Theorem 4.6.3 **(Left Invertible Iff Right Invertible)**

For $A \in M_{n\times n}(\mathbb{F})$, there exists an $n \times n$ matrix B such that $AB = I_n$ if and only if there exists an $n \times n$ matrix C such that $CA = I_n$.

Definition 4.6.4
Inverse of a Matrix

If an $n \times n$ matrix A is invertible, we refer to the matrix B such that $AB = I_n$ as the **inverse** of A. We denote the inverse of A by A^{-1}. The inverse of A satisfies

$$AA^{-1} = A^{-1}A = I_n.$$

REMARK

The above results tell us that, in order to verify that the matrix B is the inverse of A, it is sufficient to verify that $AB = I_n$. We do not need to also verify that $BA = I_n$.

Theorem 4.6.7 **(Invertibility Criteria – First Version)**

Let $A \in M_{n\times n}(\mathbb{F})$. The following three conditions are equivalent:

(a) A is invertible.

(b) $\operatorname{rank}(A) = n$.

(c) $\operatorname{RREF}(A) = I_n$.

Proposition 4.6.8 **(Algorithm for Checking Invertibility and Finding the Inverse)**

The following algorithm allows you to determine whether an $n \times n$ matrix A is invertible, and if it is, the algorithm will provide the inverse of A.

1. Construct a super-augmented matrix $[A \mid I_n]$.
2. Find the RREF, $[R \mid B]$, of $[A \mid I_n]$.
3. If $R \neq I_n$, conclude that A is not invertible. If $R = I_n$, conclude that A is invertible, and that $A^{-1} = B$.

Proposition 4.6.13 **(Inverse of a 2×2 Matrix)**

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is invertible if and only if $ad - bc \neq 0$. Furthermore, if $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Linear Transformation

5.1

Definition 5.1.1
Function Determined by a Matrix

Let $A \in M_{m\times n}(\mathbb{F})$. The **function determined by the matrix** A is the function

$$T_A \colon \mathbb{F}^n \to \mathbb{F}^m$$

defined by

$$T_A(\vec{x}) = A\vec{x}.$$

Theorem 5.1.4 **(Function Determined by a Matrix is Linear)**

Let $A \in M_{m\times n}(\mathbb{F})$ and let T_A be the function determined by the matrix A. Then T_A is linear; that is, for any $\vec{x}, \vec{y} \in \mathbb{F}^n$ and any $c \in \mathbb{F}$, the following two properties hold.

(a) $T_A(\vec{x} + \vec{y}) = T_A(\vec{x}) + T_A(\vec{y})$

(b) $T_A(c\vec{x}) = cT_A(\vec{x})$

5.2

Definition 5.2.1
Linear Transformation

We say that the function $T \colon \mathbb{F}^n \to \mathbb{F}^m$ is a **linear transformation** (or **linear mapping**) if, for any $\vec{x}, \vec{y} \in \mathbb{F}^n$ and any $c \in \mathbb{F}$, the following two properties hold.

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ (called **linearity over addition**).
2. $T(c\vec{x}) = cT(\vec{x})$ (called **linearity over scalar multiplication**).

We refer to $\mathbb{F}^n$ here as the **domain** of T and $\mathbb{F}^m$ as the **codomain** of T, as we would for any function.

Proposition 5.2.2 **(Alternate Characterization of a Linear Transformation)**

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a function. Then T is a linear transformation if and only if for any $\vec{x}, \vec{y} \in \mathbb{F}^n$ and any $c \in \mathbb{F}$,

$$T(c\vec{x} + \vec{y}) = cT(\vec{x}) + T(\vec{y}).$$

Proposition 5.2.3 **(Zero Maps to Zero)**

Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. Then

$$T(\vec{0}_{\mathbb{F}^n}) = \vec{0}_{\mathbb{F}^m}.$$

5.3

Definition 5.3.1
Range

Let $T\colon \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. We define the **range** of T, denoted $\text{Range}(T)$, to be the set of all outputs of T. That is,

$$\text{Range}(T) = \{T(\vec{x}) : \vec{x} \in \mathbb{F}^n\}.$$

The range of T is a subset of $\mathbb{F}^m$.

Proposition 5.3.2

(Range of a Linear Transformation)

Let $A \in M_{m\times n}(\mathbb{F})$, and let $T_A\colon \mathbb{F}^n \to \mathbb{F}^m$ be the linear transformation determined by A. Then

$$\text{Range}(T_A) = \text{Col}(A).$$

REMARK (Connection to Systems of Linear Equations)

We have already seen in Proposition 4.1.2 (Consistent System and Column Space) that

the system of linear equations $A\vec{x} = \vec{b}$ has a solution **if and only if** $\vec{b} \in \text{Col}(A)$.

We can now write

$A\vec{x} = \vec{b}$ is consistent **if and only if** $\vec{b} \in \text{Range}(T_A)$.

Definition 5.3.5
Onto

We say that the transformation $T\colon \mathbb{F}^n \to \mathbb{F}^m$ is **onto** (or **surjective**) if $\text{Range}(T) = \mathbb{F}^m$.

Corollary 5.3.6

(Onto Criteria)

Let $A \in M_{m\times n}(\mathbb{F})$ and let T_A be the linear transformation determined by the matrix A. The following statements are equivalent.

(a) T_A is onto.

(b) $\text{Col}(A) = \mathbb{F}^m$.

(c) $\text{rank}(A) = m$.

5.4

Definition 5.4.1
Kernel

Let $T\colon \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. We define the **kernel** of T, denoted $\operatorname{Ker}(T)$, to be the set of inputs of T whose output is zero. That is,

$$\operatorname{Ker}(T) = \left\{ \vec{x} \in \mathbb{F}^n \colon T(\vec{x}) = \vec{0}_{\mathbb{F}^m} \right\}.$$

The kernel of T is a subset of $\mathbb{F}^n$.

Proposition 5.4.2

(Kernel of a Linear Transformation)

Let $A \in M_{m\times n}(\mathbb{F})$ and let $T_A\colon \mathbb{F}^n \to \mathbb{F}^m$ be the linear transformation determined by A. Then

$$\operatorname{Ker}(T_A) = \operatorname{Null}(A).$$

Definition 5.4.3
One-to-One

We say that the transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ is **one-to-one** (or **injective**) if whenever $T(\vec{x}) = T(\vec{y})$ then $\vec{x} = \vec{y}$.

REMARK

Notice that the statement

$$\text{For all } \vec{x}, \vec{y} \in \mathbb{F}^n, \text{ if } T(\vec{x}) = T(\vec{y}) \text{ then } \vec{x} = \vec{y}$$

is logically equivalent to its contrapositive

$$\text{For all } \vec{x}, \vec{y} \in \mathbb{F}^n, \text{ if} \vec{x} \neq \vec{y} \text{ then } T(\vec{x}) \neq T(\vec{y})$$

Thus, one-to-one linear transformations have the nice property that they map distinct elements of $\mathbb{F}^n$ to distinct elements of $\mathbb{F}^m$.

Proposition 5.4.4

(One-to-One Test)

Let $T\colon \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. Then

$$T \text{ is one-to-one if and only if } \operatorname{Ker}(T) = \{\vec{0}_{\mathbb{F}^n}\}.$$

Corollary 5.4.5

(One-to-One Criteria)

Let $A \in M_{m\times n}(\mathbb{F})$ and let T_A be the linear transformation determined by the matrix A. The following statements are equivalent.

(a) T_A is one-to-one.

(b) $\operatorname{Null}(A) = \{\vec{0}_{\mathbb{F}^n}\}$.

(c) $\operatorname{nullity}(A) = 0$.

(d) $\operatorname{rank}(A) = n$.

Theorem 5.4.7 **(Invertibility Criteria – Second Version)**

Let $A \in M_{n\times n}(\mathbb{F})$ be a square matrix and let T_A be the linear transformation determined by the matrix A. The following statements are equivalent.

(a) A is invertible.

(b) T_A is one-to-one.

(c) T_A is onto.

(d) $\mathrm{Null}(A) = \{\vec{0}\}$. That is, the only solution to the homogeneous system $A\vec{x} = \vec{0}$ is the trivial solution $\vec{x} = \vec{0}$.

(e) $\mathrm{Col}(A) = \mathbb{F}^n$. That is, for every $\vec{b} \in \mathbb{F}^n$, the system $A\vec{x} = \vec{b}$ is consistent.

(f) $\mathrm{nullity}(A) = 0$.

(g) $\mathrm{rank}(A) = n$.

(h) $\mathrm{RREF}(A) = I_n$.

5.5

Example 5.5.1 Let us examine the consequences of linearity in the special case when $\mathbb{F}^n = \mathbb{F}^m = \mathbb{F}^2$. Thus suppose that $T\colon \mathbb{F}^2 \to \mathbb{F}^2$ is a linear mapping and let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a vector in $\mathbb{F}^2$. Then

$$\begin{aligned} T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \end{bmatrix}\right) \\ &= T\left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ &= x_1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \qquad \text{(by linearity)} \\ &= \left[T(\vec{e_1})\ T(\vec{e_2})\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \left[T(\vec{e_1})\ T(\vec{e_2})\right] \vec{x}. \end{aligned}$$

This shows us that the actual effect of the linear transformation can be replicated by the introduction of a matrix $\left[T(\vec{e_1})\ T(\vec{e_2})\right]$.

In addition, this matrix $\left[T(\vec{e_1})\ T(\vec{e_2})\right]$ has columns which are constructed by applying T to the basis vectors $\vec{e_1}$ and $\vec{e_2}$ in $\mathbb{F}^2$. This means that if we know what the linear transformation does to just these two (standard basis) vectors, then we can determine what it does to all vectors in $\mathbb{F}^2$.

Finally, the actual value of $T(\vec{x})$ can be computed by matrix multiplication of this matrix $\left[T(\vec{e_1})\ T(\vec{e_2})\right]$ by the component vector $\vec{x}$. This result extends to higher dimensions.

Definition 5.5.2
Standard Matrix

Let $T\colon \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. We define the **standard matrix** of T, denoted by $[T]_{\mathcal{E}}$, to be $m \times n$ matrix whose columns are the images under T of the vectors in the standard basis of $\mathbb{F}^n$:

$$[T]_{\mathcal{E}} = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \cdots & T(\vec{e_n}) \end{bmatrix}$$
$$= \begin{bmatrix} T\left(\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\right) & T\left(\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}\right) & \cdots & T\left(\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\right) \end{bmatrix}.$$

Theorem 5.5.3 **(Every Linear Transformation Is Determined by a Matrix)**

Let $T\colon \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation and let $[T]_{\mathcal{E}}$ be the standard matrix of T. Then for all $\vec{x} \in \mathbb{F}^n$,

$$T(\vec{x}) = [T]_{\mathcal{E}}\,\vec{x}$$

That is, $T = T_{[T]_{\mathcal{E}}}$ is the linear transformation determined by the matrix $[T]_{\mathcal{E}}$.

Proposition 5.5.4

Let $T\colon \mathbb{R} \to \mathbb{R}$ be a linear transformation. Then there is a real number $m \in \mathbb{R}$ such that $T(x) = mx$ for all $x \in \mathbb{R}$.

Proposition 5.5.5 **(Properties of a Standard Matrix)**

Let $A \in M_{m\times n}(\mathbb{F})$, let $T_A\colon \mathbb{F}^n \to \mathbb{F}^m$ be the linear transformation determined by A, and let $T\colon \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. Then

(a) $T_{[T]_{\mathcal{E}}} = T$.

(b) $[T_A]_{\mathcal{E}} = A$.

(c) T is onto if and only if $\operatorname{rank}([T]_{\mathcal{E}}) = m$.

(d) T is one-to-one if and only if $\operatorname{rank}([T]_{\mathcal{E}}) = n$.

5.6

$$\operatorname{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\,\vec{w}.$$

$$\text{perp}_{\vec{w}}(\vec{v}) = \vec{v} - \text{proj}_{\vec{w}}(\vec{v}).$$

$$\begin{aligned} R_\theta(\vec{x}) &= R_\theta\left(\begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix}\right) \\ &= \begin{bmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{bmatrix} \\ &= \begin{bmatrix} r(\cos\phi\cos\theta - \sin\phi\sin\theta) \\ r(\sin\phi\cos\theta + \cos\phi\sin\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta(r\cos\phi) - \sin\theta(r\sin\phi) \\ \sin\theta(r\cos\phi) + \cos\theta(r\sin\phi) \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\phi \\ r\sin\phi \end{bmatrix} \\ &= A\vec{x}, \end{aligned}$$

where $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Since we were able to express R_θ in the form of a matrix-vector product, it must be the case that R_θ is a linear transformation.

$$\text{refl}_{\vec{w}}(\vec{v}) = \vec{v} - 2\,\text{perp}_{\vec{w}}(\vec{v}).$$

5.7

Definition 5.7.1 **Composition of Linear Transformations**

Let $T_1 : \mathbb{F}^n \to \mathbb{F}^m$ and $T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be linear transformations. We define the function $T_2 \circ T_1 : \mathbb{F}^n \to \mathbb{F}^p$ by

$$(T_2 \circ T_1)(\vec{x}) = T_2(T_1(\vec{x})).$$

The function $T_2 \circ T_1$ is called the **composite function** of T_2 and T_1.

Proposition 5.7.2 **(Composition of Linear Transformations Is Linear)**

Let $T_1 : \mathbb{F}^n \to \mathbb{F}^m$ and $T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be linear transformations. Then $T_2 \circ T_1$ is a linear transformation.

Proposition 5.7.3 **(The Standard Matrix of a Composition of Linear Transformations)**

Let $T_1 : \mathbb{F}^n \to \mathbb{F}^m$ and $T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be linear transformations. Then the standard matrix of $T_2 \circ T_1$ is equal to the product of standard matrices of T_2 and T_1. That is,

$$[T_2 \circ T_1]_{\mathcal{E}} = [T_2]_{\mathcal{E}} [T_1]_{\mathcal{E}} .$$

Definition 5.7.6
Identity Transformation

The linear transformation $\text{id}_n \colon \mathbb{F}^n \to \mathbb{F}^n$ such that $\text{id}_n(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{F}^n$ is called the **identity transformation** .

EXERCISE

Show that the standard matrix $[\text{id}_n]_{\mathcal{E}}$ of id_n is the identity matrix I_n.

Definition 5.7.7
T^p

Let $T \colon \mathbb{F}^n \to \mathbb{F}^n$ and let $p > 1$ be an integer. We then define the p^{th} *power of* T, denoted by T^p, inductively by

$$T^p = T \circ T^{p-1}.$$

We also define $T^0 = \text{id}_n$.

Corollary 5.7.8

Let $T \colon \mathbb{F}^n \to \mathbb{F}^n$ be a linear transformation and let $p > 1$ be an integer. Then the standard matrix of T^p is the p^{th} power of the standard matrix of T. That is,

$$[T^p]_{\mathcal{E}} = ([T]_{\mathcal{E}})^p .$$

Determinants

6.1

Definition 6.1.1
Determinant of a 1×1 and 2×2 Matrix

If $A \in M_{1\times1}(\mathbb{F})$, then the **determinant of** A, denoted by $\det(A)$, is:

$$\det(A) = a_{11}.$$

If $A \in M_{2\times2}(\mathbb{F})$, then the **determinant of** A, denoted by $\det(A)$, is:

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Definition 6.1.4
$(i,j)^{th}$ Submatrix, $(i,j)^{th}$ minor

Let $A \in M_{n\times n}(\mathbb{F})$. The $(i,j)^{th}$ **submatrix of** A, denoted by $M_{ij}(A)$, is the $(n-1)\times(n-1)$ matrix obtained from A by removing the i^{th} row and the j^{th} column from A. The determinant of $M_{ij}(A)$ is known as the $(i,j)^{th}$ **minor of** A.

Definition 6.1.6
Determinant of an $n \times n$ matrix

Let $A \in M_{n\times n}(\mathbb{F})$ for $n \geq 2$. We define the **determinant** function, $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$, by

$$\det(A) = \sum_{j=1}^{n} a_{1j}(-1)^{1+j}\det(M_{1j}(A)).$$

Proposition 6.1.10

(i^{th} Row Expansion of the Determinant)

Let $A \in M_{n\times n}(\mathbb{F})$ with $n \geq 2$ and let $i \in \{1,\ldots,n\}$. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(M_{ij}(A)).$$

Proposition 6.1.12

(j^{th} Column Expansion of the Determinant)

Let $A \in M_{n\times n}(\mathbb{F})$ with $n \geq 2$ and let $j \in \{1,\ldots,n\}$. Then

$$\det(A) = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}\det(M_{ij}(A)).$$

Proposition 6.1.15

(Easy Determinants)

Let $A \in M_{n\times n}(\mathbb{F})$ be a square matrix.

(a) If A has a row consisting only of zeros, then $\det A = 0$.

(b) If A has a column consisting only of zeros, then $\det A = 0$.

(c) If $A = \begin{bmatrix} a_{11} & * & * & \cdots & * \\ 0 & a_{22} & * & \cdots & * \\ 0 & 0 & a_{33} & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{nn} \end{bmatrix}$ is upper triangular, then $\det A = a_{11}a_{22}\cdots a_{nn}$.

Corollary 6.1.16

The determinant of the $n \times n$ identity matrix is 1, that is, $\det(I_n) = 1$.

Proposition 6.1.17

Let $A \in M_{n\times n}(\mathbb{F})$. Then $\det(A) = \det(A^T)$.

Theorem 6.2.1 **(Effect of EROs on the Determinant)**

Let $A \in M_{n\times n}(\mathbb{F})$.

(a) (Row swap) If B is obtained from A by interchanging two rows, then $\det(B) = -\det(A)$.

(b) (Row scale) If B is obtained from A by multiplying a row by $m \neq 0$, then $\det(B) = m\det(A)$.

(c) (Row addition) If B is obtained from A by adding a non-zero multiple of one row to another row, then $\det(B) = \det(A)$.

Corollary 6.2.3 Let $A \in M_{n\times n}(\mathbb{F})$. If A has two identical rows (or two identical columns), then $\det(A) = 0$.

Corollary 6.2.4 **(Determinants of Elementary Matrices)**

For each part below, let E be an elementary matrix of the indicated type.

(a) (Row swap) $\det(E) = -1$.

(b) (Row scale) $\det(E) = m$ (if E is obtained from I_n by multiplying a row by $m \neq 0$).

(c) (Row addition) $\det(E) = 1$.

Corollary 6.2.5 **(Determinant After One ERO)**

Let $A \in M_{n\times n}(\mathbb{F})$ and suppose we perform a single ERO on A to produce the matrix B.

Assume that the corresponding elementary matrix is E. Then

$$\det(B) = \det(E)\det(A).$$

Proof: Combine Theorem 6.2.1 (Effect of EROs on the Determinant) and Corollary 6.2.4 (Determinants of Elementary Matrices). □

Corollary 6.2.6 **(Determinant After k EROs)**

Let $A \in M_{n\times n}(\mathbb{F})$ and suppose we perform a sequence of k EROs on the matrix A to obtain the matrix B.

Suppose that the elementary matrix corresponding to the ith ERO is E_i, so that

$$B = E_k \cdots E_2 E_1 A.$$

Then

$$\det(B) = \det(E_k \cdots E_2 E_1 A) = \det(E_k) \cdots \det(E_2)\det(E_1)\det(A).$$

6.3

Theorem 6.3.1 **(Invertible iff the Determinant is Non-Zero)**

Let $A \in M_{n\times n}(\mathbb{F})$. Then A is invertible if and only if $\det(A) \neq 0$.

Proposition 6.3.3 **(Determinant of a Product)**

Let $A, B \in M_{n\times n}(\mathbb{F})$. Then $\det(AB) = \det(A)\det(B)$.

Example 6.3.4 Let $A, B \in M_{n\times n}(\mathbb{F})$. Prove that AB is invertible if and only if BA is invertible.

Solution:

AB is invertible iff $\det(AB) \neq 0$ (Theorem 6.3.1 (Invertible iff the Determinant is Non-Zer

iff $\det(A)\det(B) \neq 0$ (Proposition 6.3.3 (Determinant of a Product))

iff $\det(B)\det(A) \neq 0$

iff $\det(BA) \neq 0$ (Proposition 6.3.3)

iff BA is invertible. (Theorem 6.3.1)

Corollary 6.3.5 Let $A, B \in M_{n\times n}(\mathbb{F})$. Then $\det(AB) = \det(BA)$.

Here is another useful observation that can be proved using similar ideas.

Corollary 6.3.6 **(Determinant of Inverse)**

Let $A \in M_{n\times n}(\mathbb{F})$ be invertible. Then $\det(A^{-1}) = \frac{1}{\det(A)}$.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Definition 6.4.1
Cofactor

Let $A \in M_{n\times n}(\mathbb{F})$. The $(i, j)^{th}$ **cofactor of** A, denoted by $C_{ij}(A)$, is defined by

$$C_{ij}(A) = (-1)^{i+j}\det(M_{ij}(A)).$$

Definition 6.4.2 **Adjugate of a Matrix**

Let $A \in M_{n\times n}(\mathbb{F})$. The **adjugate of** A, denoted by $\text{adj}(A)$, is the $n \times n$ matrix whose $(i,j)^{th}$ entry is

$$(\text{adj}(A))_{ij} = C_{ji}(A).$$

That is, the adjugate of A is the *transpose* of the matrix of cofactors of A.

Theorem 6.4.5

Let $A \in M_{n\times n}(\mathbb{F})$. Then

$$A\ \text{adj}(A) = \text{adj}(A)\, A = \det(A) I_n.$$

Corollary 6.4.6

(Inverse by Adjugate)

Let $A \in M_{n\times n}(\mathbb{F})$. If $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)}\,\text{adj}(A).$$

Proposition 6.5.1

(Cramer's Rule)

Let $A \in M_{n\times n}(\mathbb{F})$ and consider the equation $A\vec{x} = \vec{b}$, where $\vec{b} \in \mathbb{F}^n$ and $\det(A) \neq 0$.

If we construct B_j from A by replacing the j^{th} column of A by the column vector $\vec{b}$, then the solution $\vec{x}$ to the equation

$$A\vec{x} = \vec{b}$$

is given by

$$x_j = \frac{\det(B_j)}{\det(A)}, \quad \text{for all } j = 1, \ldots, n.$$

Example 6.5.2 Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$. Use Cramer's Rule to solve

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \vec{x} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}.$$

Solution: We saw in Example 6.2.7 that

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}\right) = -3.$$

We evaluate the following determinants using the indicated EROs.

$$\det(B_1) = \det\begin{bmatrix} -2 & 2 & 3 \\ 3 & 5 & 6 \\ -4 & 8 & 10 \end{bmatrix} = \det\begin{bmatrix} -2 & 2 & 3 \\ 0 & 8 & \frac{21}{2} \\ 0 & 4 & 4 \end{bmatrix} = 20. \qquad \begin{cases} R_2 \to \frac{3}{2}R_1 + R_2 \\ R_3 \to -2R_1 + R_3 \end{cases}$$

$$\det(B_2) = \det\begin{bmatrix} 1 & -2 & 3 \\ 4 & 3 & 6 \\ 7 & -4 & 10 \end{bmatrix} = \det\begin{bmatrix} 1 & -2 & 3 \\ 0 & 11 & -6 \\ 0 & 10 & -11 \end{bmatrix} = -61. \qquad \begin{cases} R_2 \to -4R_1 + R_2 \\ R_3 \to -7R_1 + R_3 \end{cases}$$

$$\det(B_3) = \det\begin{bmatrix} 1 & 2 & -2 \\ 4 & 5 & 3 \\ 7 & 8 & -4 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 11 \\ 0 & -6 & 10 \end{bmatrix} = 36. \qquad \begin{cases} R_2 \to -4R_1 + R_2 \\ R_3 \to -7R_1 + R_3 \end{cases}$$

Thus,

$$\vec{x} = -\frac{1}{3}\begin{bmatrix} 20 \\ -61 \\ 36 \end{bmatrix}.$$

Proposition 6.6.1 **(Area of Parallelogram)**

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ be vectors in $\mathbb{R}^2$.

The area of the parallelogram with sides $\vec{v}$ and $\vec{w}$ is $\left|\det\left(\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}\right)\right|$.

Eigen Values and Diagonalization

Definition 7.1.5

Eigenvector, Eigenvalue and Eigenpair

Let $A \in M_{n\times n}(\mathbb{F})$. A *non-zero* vector $\vec{x}$ is an **eigenvector of** A **over** $\mathbb{F}$ if there exists a scalar $\lambda \in \mathbb{F}$ such that

$$A\,\vec{x} = \lambda\,\vec{x}.$$

The scalar λ is then called an **eigenvalue of** A **over** $\mathbb{F}$, and the pair $(\lambda, \vec{x})$ is an **eigenpair of** A **over** $\mathbb{F}$.

Definition 7.2.1

Eigenvalue Equation or Eigenvalue Problem

Let $A \in M_{n\times n}(\mathbb{F})$. We refer to the equation

$$A\,\vec{x} = \lambda\,\vec{x} \quad \text{or} \quad (A - \lambda I)\,\vec{x} = \vec{0}$$

as the **eigenvalue equation for the matrix** A **over** $\mathbb{F}$. It is also sometimes referred to as the **eigenvalue problem**.

Definition 7.2.2

Characteristic Polynomial and Characteristic Equation

Let $A \in M_{n\times n}(\mathbb{F})$ and $\lambda \in \mathbb{F}$. The **characteristic polynomial of** A, denoted by $C_A(\lambda)$, is

$$C_A(\lambda) = \det(A - \lambda I).$$

The **characteristic equation of** A is

$$C_A(\lambda) = 0.$$

Proposition 7.3.1

Let $A \in M_{n\times n}(\mathbb{F})$. Then A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Definition 7.3.2

Trace

Let $A \in M_{n\times n}(\mathbb{F})$. We define the **trace** of A by

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

That is, the trace of a square matrix is the sum of its diagonal entries.

Proposition 7.3.4 **(Features of the Characteristic Polynomial)**

Let $A \in M_{n\times n}(\mathbb{F})$ have characteristic polynomial $C_A(\lambda) = \det(A - \lambda I)$. Then $C_A(\lambda)$ is a degree n polynomial in λ of the form

$$C_A(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{(n-1)} + \cdots + c_1\lambda + c_0,$$

where

(a) $c_n = (-1)^n$,

(b) $c_{n-1} = (-1)^{(n-1)}\operatorname{tr}(A)$, and

(c) $c_0 = \det(A)$.

Proposition 7.3.6 **(Characteristic Polynomial and Eigenvalues over $\mathbb{C}$)**

Let $A \in M_{n\times n}(\mathbb{F})$ have characteristic polynomial

$$C_A(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0,$$

and n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (possibly repeated) in $\mathbb{C}$. Then

(a) $c_{n-1} = (-1)^{(n-1)}\sum_{i=1}^{n}\lambda_i$, and

(b) $c_0 = \prod_{i=1}^{n}\lambda_i$.

Note that if A has repeated eigenvalues over $\mathbb{C}$, then we include each eigenvalue in the list $\lambda_1, \lambda_2, ..., \lambda_n$ as many times as its corresponding linear factor appears in the characteristic polynomial $C_A(\lambda)$.

Corollary 7.3.7 **(Eigenvalues and Trace/Determinant)**

Let $A \in M_{n\times n}(\mathbb{F})$ have n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (possibly repeated) in $\mathbb{C}$. Show that:

(a) $\sum_{i=1}^{n}\lambda_i = \operatorname{tr}(A)$.

(b) $\prod_{i=1}^{n}\lambda_i = \det(A)$.

Proposition 7.5.1 **(Linear Combinations of Eigenvectors)**

Let $c, d \in \mathbb{F}$ and suppose that $(\lambda_1, \vec{x})$ and $(\lambda_1, \vec{y})$ are eigenpairs of a matrix A over $\mathbb{F}$ with the same eigenvalue λ_1. If $c\vec{x} + d\vec{y} \neq \vec{0}$, then $(\lambda_1, c\vec{x} + d\vec{y})$ is also an eigenpair for A with eigenvalue λ_1.

Definition 7.5.3
Eigenspace

Let $A \in M_{n\times n}(\mathbb{F})$ and let $\lambda \in \mathbb{F}$. The **eigenspace of** A **associated with** λ, denoted by $E_\lambda(A)$, is the solution set to the system $(A - \lambda I)\vec{x} = \vec{0}$ over $\mathbb{F}$. That is,

$$E_\lambda(A) = \text{Null}(A - \lambda I).$$

If the choice of A is clear, we abbreviate this as E_λ.

Definition 7.6.2
Similar

Let $A, B \in M_{n\times n}(\mathbb{F})$. We say that A **is similar to** B **over** $\mathbb{F}$ if there exists an invertible matrix $P \in M_{n\times n}(\mathbb{F})$ such that $A = PBP^{-1}$.

Proposition 7.6.5

Let $A, B \in M_{n\times n}(\mathbb{F})$. If A and B are similar over $\mathbb{F}$, then they have the same characteristic polynomial and the same eigenvalues in $\mathbb{F}$.

Corollary 7.6.6

Let $A, B \in M_{n\times n}(\mathbb{F})$. If A and B are similar over $\mathbb{F}$, then:

(a) $\det(A) = \det(B)$.

(b) $\text{tr}(A) = \text{tr}(B)$.

Definition 7.6.7
Diagonalizable Matrix

Let $A \in M_{n\times n}(\mathbb{F})$. We say that A is **diagonalizable over** $\mathbb{F}$ if it is similar over $\mathbb{F}$ to a diagonal matrix $D \in M_{n\times n}(\mathbb{F})$; that is, if there exists an invertible matrix $P \in M_{n\times n}(\mathbb{F})$ such that $P^{-1}AP = D$. We say that the matrix P **diagonalizes** A.

Proposition 7.6.9

(Diagonalizable $\implies$ n Eigenvalues)

Let $A \in M_{n\times n}(\mathbb{F})$. If A is diagonalizable over $\mathbb{F}$, then the characteristic polynomial of A has n roots (possibly with repetition) in $\mathbb{F}$.

Moreover, if P diagonalizes A, then the diagonal entries of $D = P^{-1}AP$ are the eigenvalues of A.

Proposition 7.6.12

(n Distinct Eigenvalues $\implies$ Diagonalizable)

Let $A \in M_{n\times n}(\mathbb{F})$ have n *distinct* eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ in $\mathbb{F}$, let $(\lambda_1, \vec{v_1}), \ldots, (\lambda_n, \vec{v_n})$ be corresponding eigenpairs over $\mathbb{F}$, and let $P = [\vec{v_1} \cdots \vec{v_n}]$. Then

(a) P is invertible, and

(b) $P^{-1}AP = D = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

Subspaces and bases

8.1

Definition 8.1.1
Subspace

A subset V of $\mathbb{F}^n$ is called a **subspace of** $\mathbb{F}^n$ if the following properties are all satisfied.

1. $\vec{0} \in V$.
2. For all $\vec{x}$, $\vec{y} \in V$, $\vec{x} + \vec{y} \in V$ **(closure under addition)**.
3. For all $\vec{x} \in V$ and $c \in \mathbb{F}$, $c\vec{x} \in V$ **(closure under scalar multiplication)**.

Proposition 8.1.2

(Examples of Subspaces)

(a) $\{\vec{0}\}$ and $\mathbb{F}^n$ are subspaces of $\mathbb{F}^n$.

(b) If $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a subset of $\mathbb{F}^n$, then $\text{Span}\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ is a subspace of $\mathbb{F}^n$.

(c) If $A \in M_{m\times n}(\mathbb{F})$, then the solution set to the homogeneous system $A\vec{x} = \vec{0}$ is a subspace of $\mathbb{F}^n$. (Equivalently, $\text{Null}(A)$ is a subspace of $\mathbb{F}^n$.)

Proposition 8.1.3

(More Examples of Subspaces)

(a) If $A \in M_{m\times n}(\mathbb{F})$, then $\text{Col}(A)$ is a subspace of $\mathbb{F}^m$.

(b) If $T\colon \mathbb{F}^n \to \mathbb{F}^m$ is a linear transformation, then the range of T, $\text{Range}(T)$, is a subspace of $\mathbb{F}^m$.

(c) If $T\colon \mathbb{F}^n \to \mathbb{F}^m$ is a linear transformation, then the kernel of T, $\text{Ker}(T)$, is a subspace of $\mathbb{F}^n$.

(d) If $A \in M_{n\times n}(\mathbb{F})$ and if $\lambda \in \mathbb{F}$, then the eigenspace E_λ is a subspace of $\mathbb{F}^n$.

Proposition 8.1.4

(Subspace Test)

Let V be a subset of $\mathbb{F}^n$. Then V is a subspace of $\mathbb{F}^n$ **if and only if**

(a) V is non-empty, and

(b) for all $\vec{x}, \vec{y} \in V$ and $c \in \mathbb{F}$, $c\,\vec{x} + \vec{y} \in V$.

8.2

Definition 8.2.3
Linear Dependence

We say that the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{F}^n$ are **linearly dependent** if there exists scalars $c_1, c_2, \ldots, c_k \in \mathbb{F}$, not all zero, such that $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} = \vec{0}$.

If $U = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$, then we say that the set U is a **linearly dependent set** (or simply that U is **linearly dependent**) to mean that the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ are linearly dependent.

Definition 8.2.4
Linear Independence, Trivial Solution

We say that the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{F}^n$ are **linearly independent** if there do not exist scalars $c_1, c_2, \ldots, c_k \in \mathbb{F}$, not all zero, such that $c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_k\vec{v_k} = \vec{0}$.

Equivalently we say that $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k} \in \mathbb{F}^n$ are **linearly independent** if the only solution to the equation

$$c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_k\vec{v_k} = \vec{0}$$

is the **trivial solution** $c_1 = c_2 = \cdots = c_k = 0$.

If $U = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$, then we say that the set U is a **linearly independent set** (or simply that U is **linearly independent**) to mean that the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ are linearly independent.

Definition 8.2.6
Basis

Let V be a subspace of $\mathbb{F}^n$ and let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ be a finite set of vectors contained in V. We say that $\mathcal{B}$ is a **basis for** V if

1. $\mathcal{B}$ is linearly independent, and
2. $V = \operatorname{Span}(\mathcal{B})$.

Proposition 8.3.1

(Linear Dependence Check)

(a) The vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ are linearly dependent if and only if one of the vectors can be written as a linear combination of some of the other vectors.

(b) The vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$ are linearly independent if and only if

$$c_1\vec{v_1} + \cdots + c_k\vec{v_k} = \vec{0} \quad (c_i \in \mathbb{F}) \quad \text{implies} \quad c_1 = \cdots = c_k = 0.$$

Proposition 8.3.2

Let $S \subseteq \mathbb{F}^n$.

(a) If $\vec{0} \in S$, then S is linearly dependent.

(b) If $S = \{\vec{x}\}$ contains only one vector, then S is linearly dependent if and only if $\vec{x} = \vec{0}$.

(c) If $S = \{\vec{x}, \vec{y}\}$ contains only two vectors, then S is linearly dependent if and only if one of the vectors is a multiple of the other.

Proposition 8.3.6

(Pivots and Linear Independence)

Let $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{F}^n$. Let $A = [\vec{v_1}\ \vec{v_2}\ \cdots\ \vec{v_k}]$ be the $n \times k$ matrix whose columns are the vectors in S.

Suppose that $\text{rank}(A) = r$ and A has pivots in columns $q_1, q_2, \ldots, q_r$.

Let $U = \{\overrightarrow{v_{q_1}}, \overrightarrow{v_{q_2}}, \ldots, \overrightarrow{v_{q_r}}\}$, the set of columns of A that correspond to the pivot columns labelled above. Then

(a) S is linearly independent if and only if $r = k$.

(b) U is linearly independent.

(c) If $\vec{v}$ is in S but not in U then the set $\{\overrightarrow{v_{q_1}}, ..., \overrightarrow{v_{q_r}}, \vec{v}\}$ is linearly dependent.

(d) $\text{Span}\,(U) = \text{Span}\,(S)$.

Corollary 8.3.7

(Bound on Number of Linearly Independent Vectors)

Let $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{F}^n$. If $n < k$, then S is linearly dependent.

8.4 Spanning Set

Theorem 8.4.1

(Every Subspace Has a Spanning Set)

Let V be a subspace of $\mathbb{F}^n$. Then there exist vectors $\vec{v_1}, \ldots, \vec{v_k} \in V$ such that

$$V = \text{Span}\{\vec{v_1}, \ldots, \vec{v_k}\}.$$

Proposition 8.4.2 **(Span of Subset)**

Let V be a subspace of $\mathbb{F}^n$ and let $S = \{\vec{v_1}, \ldots, \vec{v_k}\} \subseteq V$. Then $\operatorname{Span}(S) \subseteq V$.

Proposition 8.4.6 **(Spans $\mathbb{F}^n$ iff Rank is n)**

Let $S = \{\vec{v_1}, \ldots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{F}^n$ and let $A = \begin{bmatrix} \vec{v_1} & \cdots & \vec{v_k} \end{bmatrix}$ be the matrix whose columns are the vectors in S. Then

$$\operatorname{Span}(S) = \mathbb{F}^n \quad \text{if and only if} \quad \operatorname{rank}(A) = n.$$

8.5 Basis

Theorem 8.5.1 **(Every Subspace Has a Basis)**

Let V be a subspace of $\mathbb{F}^n$. Then V has a basis.

Definition 8.5.2
Standard Basis for $\mathbb{F}^n$

In $\mathbb{F}^n$, let $\vec{e}_i$ represent the vector whose i^{th} component is 1 with all other components 0. The set $\mathcal{E} = \{\vec{e_1}, \vec{e_2}, ..., \vec{e_n}\}$ is called the **standard basis for** $\mathbb{F}^n$.

Proposition 8.5.3 **(Size of Basis for $\mathbb{F}^n$)**

Let $S = \{\vec{v_1}, \ldots, \vec{v_k}\}$ be a set of k vectors in $\mathbb{F}^n$. If S is a basis for $\mathbb{F}^n$, then $k = n$.

Proposition 8.5.4 **(n Vectors in $\mathbb{F}^n$ Span iff Independent)**

Let $S = \{\vec{v_1}, \ldots, \vec{v_n}\}$ be a set of n vectors in $\mathbb{F}^n$. Then S is linearly independent if and only if $\operatorname{Span}(S) = \mathbb{F}^n$.

REMARK

There are two problems we might encounter when trying to obtain a basis for $\mathbb{F}^n$:

(a) We might have a set of vectors $S \subseteq \mathbb{F}^n$ with the property that $\operatorname{Span}(S) = \mathbb{F}^n$, but the set contains more than n vectors which is too many to be a basis. In this case, S will be linearly dependent. We may apply Proposition 8.3.6 (Pivots and Linear Independence) to produce a subset of S that is linearly independent, but still spans $\mathbb{F}^n$. This subset will be a basis for $\mathbb{F}^n$.

(b) We might have a set of vectors $S \subseteq \mathbb{F}^n$ that is linearly independent, but that contains fewer than n vectors which is too few to be a basis. In this case, $\operatorname{Span}(S) \neq \mathbb{F}^n$. The problem here is to figure out which vectors to add to S to make it span $\mathbb{F}^n$. One possible approach is to add all n standard basis vectors to S, obtaining a larger set S'. Then certainly $\operatorname{Span}(S') = \mathbb{F}^n$, but now S' is too large to be a basis. This brings us back to **(a)**.

8.6 basis for col(A) and Null(A)

Proposition 8.6.1 **(Basis for** $\operatorname{Col}(A)$**)**

Let $A = [\vec{a_1} \cdots \vec{a_n}] \in M_{m\times n}(\mathbb{F})$ and suppose that $\operatorname{RREF}(A)$ has pivots in columns $q_1, \ldots, q_r$, where $r = \operatorname{rank}(A)$. Then $\{\vec{a_{q_1}}, \ldots, \vec{a_{q_r}}\}$ is a basis for $\operatorname{Col}(A)$.

Proposition 8.6.5 **(Basis for** $\operatorname{Null}(A)$**)**

Let $A \in M_{m\times n}(\mathbb{F})$ and consider the homogeneous linear system $A\vec{x} = \vec{0}$. Suppose that, after applying the Gauss–Jordan Algorithm, we obtain k free parameters so that the solution set to this system is given by

$$\operatorname{Null}(A) = \{t_1\vec{x_1} + \cdots + t_k\vec{x_k} \colon t_1, \ldots, t_k \in \mathbb{F}\}.$$

Here $k = \operatorname{nullity}(A) = n - \operatorname{rank}(A)$ and the parameters t_i and the vectors $\vec{x_i}$ for $1 \leq i \leq k$ are obtained using the method outlined in Section 3.7.

Then $\{\vec{x_1}, \ldots, \vec{x_k}\}$ is a basis for $\operatorname{Null}(A)$.

8.7 Dimension

Theorem 8.7.2 **(Dimension is Well-Defined)**

Let V be a subspace of $\mathbb{F}^n$. If $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_k}\}$ and $\mathcal{C} = \{\vec{w_1}, \ldots, \vec{w_\ell}\}$ are bases for V, then $k = \ell$.

Definition 8.7.3 **Dimension**

The number of elements in a basis for a subspace V of $\mathbb{F}^n$ is called the **dimension** of V. We denote this number by $\dim(V)$.

Proposition 8.7.5 **(Bound on Dimension of Subspace)**

Let V be a subspace of $\mathbb{F}^n$. Then $\dim(V) \leq n$.

Proposition 8.7.8 **(Rank and Nullity as Dimensions)**

Let $A \in M_{m\times n}(\mathbb{F})$. Then

(a) $\text{rank}(A) = \dim(\text{Col}(A))$, and

(b) $\text{nullity}(A) = \dim(\text{Null}(A))$.

Theorem 8.7.9 **(Rank–Nullity Theorem)**

Let $A \in M_{m\times n}(\mathbb{F})$. Then

$$\begin{aligned} n &= \text{rank}(A) + \text{nullity}(A) \\ &= \dim(\text{Col}(A)) + \dim(\text{Null}(A)). \end{aligned}$$

8.8 Coordinates

Theorem 8.8.1 **(Unique Representation Theorem)**

Let V be a subspace of $\mathbb{F}^n$ and let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ be a basis for V. Then, for every vector $\vec{v} \in V$, there exist *unique* scalars $c_1, c_2, \ldots, c_k \in \mathbb{F}$ such that

$$\vec{v} = c_1\,\vec{v_1} + c_2\,\vec{v_2} + \cdots + c_k\,\vec{v_k}.$$

Definition 8.8.3
Coordinates and Components

Let V be a subspace of $\mathbb{F}^n$ and let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ be a basis for V. Let the vector $\vec{v} \in V$ have representation

$$\vec{v} = c_1\,\vec{v_1} + c_2\,\vec{v_2} + \cdots + c_k\,\vec{v_k} = \sum_{i=1}^{k} c_i\,\vec{v_i}, \qquad (c_i \in \mathbb{F}).$$

We call the scalars $c_1, c_2, \ldots, c_k$ the **coordinates** (or **components**) **of $\vec{v}$ with respect to $\mathcal{B}$**, or the **$\mathcal{B}$-coordinates of** $\vec{v}$.

Definition 8.8.4
Ordered Basis

Let V be a subspace of $\mathbb{F}^n$. An **ordered basis for** V is a basis $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ for V together with a fixed ordering.

Definition 8.8.6
Coordinate Vector

Let $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_k}\}$ be an ordered basis for the subspace V of $\mathbb{F}^n$. Let $\vec{v} \in V$ have coordinates $c_1, \ldots, c_k$ with respect to $\mathcal{B}$, where the ordering of the scalars c_i matches the ordering in $\mathcal{B}$, that is,

$$\vec{v} = \sum_{i=1}^{k} c_i \vec{v_i}.$$

Then the **coordinate vector of $\vec{v}$ with respect to $\mathcal{B}$** (or the **$\mathcal{B}$-coordinate vector of**

$\vec{v}$) is the column vector in $\mathbb{F}^n$

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}.$$

Theorem 8.8.8

(Linearity of Taking Coordinates)

Let $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_k}\}$ be an ordered basis for V. Then the function $[\ ]_{\mathcal{B}} : V \to \mathbb{F}^k$ given by $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a linear transformation.

Definition 8.8.12

Change-of-Basis Matrix, Change-of-Coordinate Matrix

Let $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_k}\}$ and $\mathcal{C} = \{\vec{w_1}, \ldots, \vec{w_k}\}$ be ordered bases for a subspace V of $\mathbb{F}^n$.

The **change-of-basis** (or **change-of-coordinates**) matrix **from $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates** is the $k \times k$ matrix

$$_{\mathcal{C}}[I]_{\mathcal{B}} = \left[\, [\vec{v_1}]_{\mathcal{C}}, \ldots, [\vec{v_k}]_{\mathcal{C}} \,\right]$$

whose columns are the $\mathcal{C}$-coordinates of the vectors $\vec{v_i}$ in $\mathcal{B}$.

Similarly, the **change-of-basis** (or **change-of-coordinates**) matrix **from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates** is the $k \times k$ matrix

$$_{\mathcal{B}}[I]_{\mathcal{C}} = \left[\, [\vec{w_1}]_{\mathcal{B}}, \ldots, [\vec{w_k}]_{\mathcal{B}} \,\right]$$

whose columns are the $\mathcal{B}$-coordinates of the vectors $\vec{w_i}$ in $\mathcal{C}$.

Proposition 8.8.14

(Changing a Basis)

Let $\mathcal{B} = \{\vec{v_1}, \ldots, \vec{v_k}\}$ and $\mathcal{C} = \{\vec{w_1}, \ldots, \vec{w_k}\}$ be ordered bases for a subspace V of $\mathbb{F}^n$.

Then $[\vec{x}]_{\mathcal{C}} = {}_{\mathcal{C}}[I]_{\mathcal{B}}\, [\vec{x}]_{\mathcal{B}}$ and $[\vec{x}]_{\mathcal{B}} = {}_{\mathcal{B}}[I]_{\mathcal{C}}\, [\vec{x}]_{\mathcal{C}}$ for all $\vec{x} \in V$.

Corollary 8.8.15

Let $\vec{x} = [\vec{x}]_{\mathcal{E}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be a vector in $\mathbb{F}^n$, where $\mathcal{E}$ is the standard basis for $\mathbb{F}^n$. If $\mathcal{C}$ is any ordered basis for $\mathbb{F}^n$, then

$$[\vec{x}]_{\mathcal{C}} = {}_{\mathcal{C}}[I]_{\mathcal{E}}\, [\vec{x}]_{\mathcal{E}}.$$

Corollary 8.8.16

(Inverse of Change-of-Basis Matrix)

Let $\mathcal{B}$ and $\mathcal{C}$ be two ordered bases of $\mathbb{F}^n$. Then

$$_{\mathcal{B}}[I]_{\mathcal{C}}\; {}_{\mathcal{C}}[I]_{\mathcal{B}} = I_n \qquad \text{and} \qquad {}_{\mathcal{C}}[I]_{\mathcal{B}}\; {}_{\mathcal{B}}[I]_{\mathcal{C}} = I_n.$$

In other words, $_{\mathcal{B}}[I]_{\mathcal{C}} = \left({}_{\mathcal{C}}[I]_{\mathcal{B}} \right)^{-1}$ and $_{\mathcal{C}}[I]_{\mathcal{B}} = \left({}_{\mathcal{B}}[I]_{\mathcal{C}} \right)^{-1}$.

Chapter 9

Diagonalization

Definition 9.1.1
$\mathcal{B}$-Matrix of T

Let $T\colon \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator and let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ be an ordered basis for $\mathbb{F}^n$. We define the **$\mathcal{B}$-matrix of** T to be the matrix $[T]_\mathcal{B}$ constructed as follows.

$$[T]_\mathcal{B} = \Big[[T(\vec{v_1})]_\mathcal{B} \quad [T(\vec{v_2})]_\mathcal{B} \cdots [T(\vec{v_n})]_\mathcal{B}\Big]$$

That is, after applying the action of T to each member of $\mathcal{B}$, we take the $\mathcal{B}$-coordinate vectors of each of these images to create the columns of $[T]_\mathcal{B}$.

Proposition 9.1.2

Let $T\colon \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator and let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ be an ordered basis for $\mathbb{F}^n$. If $\vec{v} \in \mathbb{F}^n$, then

$$[T(\vec{v})]_\mathcal{B} = [T]_\mathcal{B}\ [\vec{v}]_\mathcal{B}.$$

Proposition 9.1.5

(Similarity of Matrix Representations)

Let $T\colon \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. Let $\mathcal{B}$ and $\mathcal{C}$ be ordered bases for $\mathbb{F}^n$. Then

$$[T]_\mathcal{C} = {}_\mathcal{C}[I]_\mathcal{B}\ [T]_\mathcal{B}\ {}_\mathcal{B}[I]_\mathcal{C} = ({}_\mathcal{B}[I]_\mathcal{C})^{-1}\ [T]_\mathcal{B}\ {}_\mathcal{B}[I]_\mathcal{C}$$

and

$$[T]_\mathcal{B} = {}_\mathcal{B}[I]_\mathcal{C}\ [T]_\mathcal{C}\ {}_\mathcal{C}[I]_\mathcal{B} = ({}_\mathcal{C}[I]_\mathcal{B})^{-1}\ [T]_\mathcal{C}\ {}_\mathcal{C}[I]_\mathcal{B}.$$

That is, the matrices $[T]_\mathcal{B}$ and $[T]_\mathcal{C}$ are similar over $\mathbb{F}$.

Corollary 9.1.6

(Finding the Standard Matrix)

Let $T\colon \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. Let $\mathcal{B}$ be a basis for $\mathbb{F}^n$ and let $\mathcal{E}$ be the standard basis for $\mathbb{F}^n$. Then

$$[T]_\mathcal{E} = {}_\mathcal{E}[I]_\mathcal{B}\ [T]_\mathcal{B}\ {}_\mathcal{B}[I]_\mathcal{E} = ({}_\mathcal{B}[I]_\mathcal{E})^{-1}\ [T]_\mathcal{B}\ {}_\mathcal{B}[I]_\mathcal{E}$$

and

$$[T]_\mathcal{B} = {}_\mathcal{B}[I]_\mathcal{E}\ [T]_\mathcal{E}\ {}_\mathcal{E}[I]_\mathcal{B} = ({}_\mathcal{E}[I]_\mathcal{B})^{-1}\ [T]_\mathcal{E}\ {}_\mathcal{E}[I]_\mathcal{B}.$$

9.2

Definition 9.2.1
Eigenvector, Eigenvalue and Eigenpair of a Linear Operator

Let $T\colon \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. We say that the *non-zero* vector $\vec{x} \in \mathbb{F}^n$ is an **eigenvector** of T to mean that there exists a scalar $\lambda \in \mathbb{F}$ such that

$$T(\vec{x}) = \lambda\vec{x}.$$

This equation is called the **eigenvalue equation** or the **eigenvalue problem**. The scalar λ is called an **eigenvalue** of T and the pair $(\lambda, \vec{x})$ is called an **eigenpair** of T.

Proposition 9.2.2 **(Eigenpairs of T and $[T]_{\mathcal{B}}$)**

Let $T\colon \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator and let $\mathcal{B}$ be an ordered basis of $\mathbb{F}^n$. Then $(\lambda, \vec{x})$ is an eigenpair of T if and only if $(\lambda, [\vec{x}]_{\mathcal{B}})$ is an eigenpair of the matrix $[T]_{\mathcal{B}}$.

Definition 9.2.4 **Diagonalizable**

Let $T\colon \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. We say that T is **diagonalizable over** $\mathbb{F}$ to mean that there exists an ordered basis $\mathcal{B}$ of $\mathbb{F}^n$ such that $[T]_{\mathcal{B}}$ is a diagonal matrix.

Proposition 9.2.5 **(Eigenvector Basis Criterion for Diagonalizability)**

Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator. Then T is diagonalizable over $\mathbb{F}$ if and only if there exists an ordered basis $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ of $\mathbb{F}^n$ consisting of eigenvectors of T.

Proposition 9.2.7 **(T Diagonalizable iff $[T]_{\mathcal{B}}$ Diagonalizable)**

Let $T\colon \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator and let $\mathcal{B}$ be an ordered basis of $\mathbb{F}^n$. Then T is diagonalizable over $\mathbb{F}$ if and only if the matrix $[T]_{\mathcal{B}}$ is diagonalizable over $\mathbb{F}$.

Corollary 9.2.8 **(Eigenvector Basis Criterion for Diagonalizability – Matrix Version)**

Let $A \in M_{n\times n}(\mathbb{F})$. Then A is diagonalizable over $\mathbb{F}$ if and only if there exists a basis of $\mathbb{F}^n$ consisting of eigenvectors of A.

special case where A has n distinct eigenvalues in $\mathbb{F}$. To do so, we need the following result.

Proposition 9.2.10 **(Eigenvectors Corresponding to Distinct Eigenvalues are Linearly Independent)**

Let $A \in M_{n\times n}(\mathbb{F})$ have eigenpairs $(\lambda_1, \vec{v_1}), (\lambda_2, \vec{v_2}), \dots, (\lambda_k, \vec{v_k})$, for $1 \le k \le n$.

If the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ are all distinct, then the set of eigenvectors $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$ is linearly independent.

Let $A \in M_{n\times n}(\mathbb{F})$ have n *distinct* eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ in $\mathbb{F}$, let $(\lambda_1, \vec{v_1}), \dots, (\lambda_n, \vec{v_n})$ be corresponding eigenpairs over $\mathbb{F}$, and let $P = [\vec{v_1} \cdots \vec{v_n}]$. Then

(a) P is invertible, and

(b) $P^{-1}AP = D = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

Definition 9.2.11
Characteristic Polynomial

Let $T\colon \mathbb{F}^n \to \mathbb{F}^n$ be a linear operator and let $\mathcal{B}$ be a basis for $\mathbb{F}^n$. The **characteristic polynomial** of T, $C_T(\lambda)$, is the characteristic polynomial of the matrix $[T]_{\mathcal{B}}$:

$$C_T(\lambda) = C_{[T]_{\mathcal{B}}}(\lambda).$$

Definition 9.2.16
Algebraic Multiplicity

Let λ_i be an eigenvalue of $A \in M_{n\times n}(\mathbb{F})$. The **algebraic multiplicity** of λ_i, denoted by a_{λ_i}, is the largest positive integer such that $(\lambda - \lambda_i)^{a_{\lambda_i}}$ divides the characteristic polynomial $C_A(\lambda)$.

In other words, a_{λ_i} gives the number of times that $(\lambda - \lambda_i)$ terms occur in the fully factorized form of $C_A(\lambda)$.

Definition 9.2.18
Geometric Multiplicity

Let λ_i be an eigenvalue of $A \in M_{n\times n}(\mathbb{F})$. The **geometric multiplicity** of λ_i, denoted by g_{λ_i}, is the dimension of the eigenspace E_{λ_i}. That is, $g_{\lambda_i} = \dim(E_{\lambda_i})$.

Proposition 9.2.20

(Geometric and Algebraic Multiplicities)

Let λ_i be an eigenvalue of the matrix $A \in M_{n\times n}(\mathbb{F})$. Then

$$1 \le g_{\lambda_i} \le a_{\lambda_i}.$$

Proposition 9.2.21

Let $A \in M_{n\times n}(\mathbb{F})$ with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. If their corresponding eigenspaces, $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_k}$ have bases $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ is linearly independent.

Theorem 9.2.22

(Diagonalizability Test)

Let $A \in M_{n\times n}(\mathbb{F})$ with characteristic polynomial

$$C_A(\lambda) = (\lambda - \lambda_1)^{a_{\lambda_1}} \cdots (\lambda - \lambda_k)^{a_{\lambda_k}}\, h(\lambda),$$

where $\lambda_1, \ldots \lambda_k$ are all of the distinct eigenvalues of A in $\mathbb{F}$ with corresponding algebraic multiplicities $a_{\lambda_1} \ldots a_{\lambda_k}$ and $h(\lambda)$ is a polynomial in λ that is irreducible over $\mathbb{F}$. Then A is diagonalizable over $\mathbb{F}$ if and only $h(\lambda)$ is a constant polynomial and $a_{\lambda_i} = g_{\lambda_i}$, for each $i = 1, \ldots, k$.

Proposition 9.3.1 **(Powers of Similar Matrices)**

Let $A, B \in M_{n\times n}(\mathbb{F})$ such that $B = P^{-1}AP$ for some invertible matrix $P \in M_{n\times n}(\mathbb{F})$, so that A and B are similar. Then

$$B^k = P^{-1}A^kP.$$

Vector Spaces

We can think of linear algebra as operating in a world with four components.

1. A non-empty set of objects, $\mathbb{V}$.
2. A field, $\mathbb{F}$.
3. An operation, called **addition**, that combines two objects from $\mathbb{V}$, which we denote by $\oplus$.
4. An operation, called **scalar multiplication**, which combines an object from $\mathbb{V}$ and a scalar from $\mathbb{F}$, which we denote by $\odot$.

Definition 10.2.1
Vector Space

A non-empty set of objects, $\mathbb{V}$, is a **vector space over a field, $\mathbb{F}$, under the operations of addition, $\oplus$, and scalar multiplication, $\odot$,** provided the following set of ten axioms are met.

C1. For all $\vec{x}, \vec{y} \in \mathbb{V}$, $\vec{x} \oplus \vec{y} \in \mathbb{V}$.
(Closure under Addition)

C2. For all $\vec{x} \in \mathbb{V}$ and all $c \in \mathbb{F}$, $c \odot \vec{x} \in \mathbb{V}$.
(Closure under Scalar Multiplication)

V1. For all $\vec{x}, \vec{y} \in \mathbb{V}$, $\vec{x} \oplus \vec{y} = \vec{y} \oplus \vec{x}$.
(Addition is Commutative)

V2. For all $\vec{x}, \vec{y}, \vec{z} \in \mathbb{V}$, $(\vec{x} \oplus \vec{y}) \oplus \vec{z} = \vec{x} \oplus (\vec{y} \oplus \vec{z}) = \vec{x} \oplus \vec{y} \oplus \vec{z}$.
(Addition is Associative)

V3. There exists a vector $\vec{0} \in \mathbb{V}$ such that for all $\vec{x} \in \mathbb{V}$, $\vec{x} \oplus \vec{0} = \vec{0} \oplus \vec{x} = \vec{x}$.
(Additive Identity)

V4. For all $\vec{x} \in \mathbb{V}$, there exists a vector $-\vec{x} \in \mathbb{V}$ such that $\vec{x} \oplus (-\vec{x}) = (-\vec{x}) \oplus \vec{x} = \vec{0}$.
(Additive Inverse)

V5. For all $\vec{x}, \vec{y} \in \mathbb{V}$ and for all $c \in \mathbb{F}$, $c \odot (\vec{x} \oplus \vec{y}) = (c \odot \vec{x}) \oplus (c \odot \vec{y})$.
(Vector Addition Distributive Law)

V6. For all $\vec{x} \in \mathbb{V}$ and for all $c, d \in \mathbb{F}$, $(c + d) \odot \vec{x} = (c \odot \vec{x}) \oplus (d \odot \vec{x})$.
(Scalar Addition Distributive Law)

V7. For all $\vec{x} \in \mathbb{V}$ and for all $c, d \in \mathbb{F}$, $(cd) \odot \vec{x} = c \odot (d \odot \vec{x})$.
(Scalar Multiplication is Associative)

V8. For all $\vec{x} \in \mathbb{V}$, $1 \odot \vec{x} = \vec{x}$.
(Multiplicative Identity)

Definition 10.2.2
Vector

A **vector** is an element of a vector space.

Definition 10.2.6
$L(\mathbb{F}^n, \mathbb{F}^m)$

We use $L(\mathbb{F}^n, \mathbb{F}^m)$ to denote the vector space over $\mathbb{F}$ comprised of all linear transformations $T : \mathbb{F}^n \to \mathbb{F}^m$, with the following addition and scalar multiplication operations for all $x \in \mathbb{F}^n$ and all $c \in \mathbb{F}$ as follows:

$$(T_1 + T_2)(\vec{x}) = T_1(\vec{x}) + T_2(\vec{x}),$$
$$(cT)(\vec{x}) = cT(\vec{x}).$$

Proposition 10.3.1 Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. The zero vector in $\mathbb{V}$ is unique.

Proposition 10.3.2 Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $\vec{x} \in \mathbb{V}$. The additive inverse of $\vec{x}$ is unique.

Proposition 10.3.3 Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and $\vec{x} \in \mathbb{V}$. Then

(a) For all $\vec{x} \in \mathbb{V}$, $0 \odot \vec{x} = \vec{0}$, and

(b) For all $a \in \mathbb{F}$, $a \odot \vec{0} = \vec{0}$.

Proposition 10.3.4 Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $\vec{x} \in \mathbb{V}$. Then

$$-\vec{x} = (-1) \odot \vec{x}.$$

Proposition 10.3.5 **(Cancellation Law)**

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $\vec{x} \in \mathbb{V}$ and $a \in \mathbb{F}$.

$$\text{If } a \odot \vec{x} = \vec{0}, \text{ then } a = 0 \text{ or } \vec{x} = \vec{0}.$$

Definition 10.4.1
Linear Combination

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v}_k \in \mathbb{V}$ and let $c_1, c_2, \ldots, c_k \in \mathbb{F}$. We refer to

$$(c_1 \odot \vec{v_1}) \oplus (c_2 \odot \vec{v_2}) \oplus \cdots \oplus (c_k \odot \vec{v}_k)$$

as a **linear combination** of $\vec{v_1}, \vec{v_2}, \ldots, \vec{v}_k$.

Definition 10.4.4
Span

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and let $W = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v}_k\} \subseteq \mathbb{V}$. The **span** of W is the set of all linear combinations of elements of W. That is,

$$\text{Span}\,(W) = \{(c_1 \odot \vec{v_1}) \oplus (c_2 \odot \vec{v_2}) \oplus \cdots \oplus (c_k \odot \vec{v}_k) \colon c_i \in \mathbb{F},\ i = 1, \ldots, k\}.$$

Definition 10.4.9
Subspace

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and let $\mathbb{U}$ be a non-empty subset of $\mathbb{V}$. We say that $\mathbb{U}$ is a **subspace** of $\mathbb{V}$ if $\mathbb{U}$ is a vector space over $\mathbb{F}$ using the same addition and scalar multiplication operations as $\mathbb{V}$.

Theorem 10.4.10 **(Subspace Test)**

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and let $\mathbb{U}$ be a subset of $\mathbb{V}$. Then $\mathbb{U}$ is a subspace of $\mathbb{V}$ if and only if all of the following conditions hold:

1. $\mathbb{U}$ is non-empty,

2. $\mathbb{U}$ is closed under addition (C1), and

3. $\mathbb{U}$ is closed under scalar multiplication (C2).

Proposition 10.4.12 Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Let $W = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \ldots, \overrightarrow{w_n}\} \subseteq \mathbb{V}$. Then

(a) $\operatorname{Span}(W)$ is a subspace of $\mathbb{V}$.

(b) If $\mathbb{U}$ is a subspace of $\mathbb{V}$ such that $W \subseteq \mathbb{U}$, then $\operatorname{Span}(W) \subseteq \mathbb{U}$.

Definition 10.4.15
Linearly Independent, Linearly Dependent

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and let $W = \{\overrightarrow{w_1}, \overrightarrow{w_2}, \ldots, \overrightarrow{w_n}\} \subseteq \mathbb{V}$. We say that W is **linearly independent** if the only solution to the equation

$$(a_1 \odot \overrightarrow{w_1}) \oplus (a_2 \odot \overrightarrow{w_2}) \oplus \cdots \oplus (a_n \odot \overrightarrow{w_n}) = \vec{0}$$

is the trivial solution, $a_1 = a_2 = \cdots = a_n = 0$. Otherwise, we say that W is **linearly dependent**.

Definition 10.4.17
Basis

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and let $\mathcal{B} = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}\} \subseteq \mathbb{V}$. We say that $\mathcal{B}$ is a **basis** for $\mathbb{V}$ if $\mathcal{B}$ is linearly independent and if $\operatorname{Span}(\mathcal{B}) = \mathbb{V}$.

The basis for the zero vector space, $\{\vec{0}\}$, is defined to be the empty set $\varnothing$.

Definition 10.4.20
Dimension, Infinite Dimensional

If $\mathcal{B} = \{\overrightarrow{v_1}, \ldots, \overrightarrow{v_n}\}$ is a basis for a vector space $\mathbb{V}$ over $\mathbb{F}$, then we say the **dimension** of $\mathbb{V}$ is n. We denote this by writing $\dim(V) = n$.

The **dimension** of the zero vector space $\{\vec{0}\}$ is 0.

If $\mathbb{V}$ does not have a basis with a finite number of vectors in it, then $\mathbb{V}$ is said to be **infinite-dimensional**.

Theorem 10.4.23 **(Unique Representation Theorem)**

Let $\mathcal{B} = \{\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}\}$ be a basis for a vector space $\mathbb{V}$ over $\mathbb{F}$. Then for every vector $\vec{v} \in \mathbb{F}^n$, there exist unique scalars $c_1, c_2, \ldots, c_n \in \mathbb{F}$ such that $\vec{v} = c_1 \overrightarrow{v_1} + c_2 \overrightarrow{v_2} + \cdots + c_n \overrightarrow{v_n}$.

Definition 10.4.24

$\mathcal{B}$-Coordinates, $\mathcal{B}$-Coordinate Vector

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ be an ordered basis for $\mathbb{V}$. Let

$$\vec{v} = (c_1 \odot \vec{v_1}) \oplus (c_2 \odot \vec{v_2}) \oplus \cdots \oplus (c_n \odot \vec{v_n})$$

be the unique representation of $\vec{v}$ as a linear combination of the vectors in $\mathcal{B}$. The scalars $c_1, c_2, \ldots, c_n$ are referred to as the **$\mathcal{B}$-coordinates** of $\vec{v}$ and the vector

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

is known as the **$\mathcal{B}$-coordinate vector** or the **coordinate vector of $\vec{v}$ with respect to the basis $\mathcal{B}$**.

Definition 10.4.28

Change-of-Basis Matrix

Let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ and $\mathcal{C} = \{\vec{w_1}, \vec{w_2}, \ldots, \vec{w_n}\}$ be two ordered bases for the vector space $\mathbb{V}$ over $\mathbb{F}$. The **change-of-basis matrix** from $\mathcal{B}$ to $\mathcal{C}$, denoted by ${}_{\mathcal{C}}[I]_{\mathcal{B}}$, is the matrix

$${}_{\mathcal{C}}[I]_{\mathcal{B}} = \left[[\vec{v_1}]_{\mathcal{C}}, [\vec{v_2}]_{\mathcal{C}}, \ldots, [\vec{v_n}]_{\mathcal{C}} \right].$$

Proposition 10.4.29

Let $\mathcal{B} = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ and $\mathcal{C} = \{\vec{w_1}, \vec{w_2}, \ldots, \vec{w_n}\}$ be two ordered bases for the vector space $\mathbb{V}$ over $\mathbb{F}$. Then

$${}_{\mathcal{B}}[I]_{\mathcal{C}} = \left({}_{\mathcal{C}}[I]_{\mathcal{B}}\right)^{-1}.$$